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M.Sc. (Sem.-I) Examination November - 2025

Mathematics : Ordinary Differential Equations (Paper : PGMTH-102)

Time: 3 Hours] **[Total Marks: 70**

સૂચના : / Instructions	Seat No.:
(1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fill up strictly the above details on your answer book (2) Answer all questions. (3) Figures to the right indicates marks. (4) Follow usual notations and conventions.	Student's Signature

Q.1. Attempt Any Two: **14**

(a) Let the functions $b_1, b_2, \ldots \ldots . . b_n$ in equation
$L(x) = x^{(n)}(t) + b_1(t)\, x^{(n-1)}(t) + \cdots..+ b_n(t)\, x(t) = 0,\ t \in I$ be
defined and continuous on an interval I. Let $\varphi_1, \varphi_2, \ldots \ldots . . \varphi_n$ be n
linearly independent solutions of
$L(x) = x^{(n)}(t) + b_1(t)\, x^{(n-1)}(t) + \cdots.. + b_n(t)\, x(t) = 0,\ t \in I$ existing
on I containing a point t_0. Then prove that Wronskian

$$W(t) = exp\left[-\int_{t_0}^{t} b_1(s)\,ds\right] W(t_0),;\ t_0, t \in I$$

(b) Solve by using method of Undetermined co-efficient:
$x^{(4)} + 4x = 2\sin t + 1 + 3t^2 + 4e^t$

(c) Solve by using method of variation of parameters:
$x''' - x' = cost$

Q.2. Attempt Any Two: **14**

(a) If $P_n(t)$ is a Legendre polynomial, then show that $\int_{-1}^{1} P_n^2(t)\,dt = \frac{2}{2n+1}$

(b) Consider the equation $2t^2 x'' + t(2t+1)\,x' - x = 0$.
Prove that the general solution of the equation is given by
$x(t) = C_1 t\left[1 - \frac{2}{5}t + \frac{4}{35}t^2 - \cdots\right] + C_2 t^{-1/2}\left[1 - t + \frac{1}{2}t^2 - \cdots\right]$

(c) Prove that: (i) $\frac{d}{dt}[t^p J_p(t)] = t^p J_{p-1}(t)$ and

(ii) $\frac{d}{dt}[t^{-p} J_p(t)] = -t^{-p} J_{p+1}(t)$

Q.3. Attempt Any Two: **14**

(a) Prove that a solution matrix Φ of $X' = A(t)X,\ t \in I$ on I is a fundamental matrix of $x' = A(t)x$ on I if and only if $det\ \Phi(t) \neq o,\ t \in I$.

(b) Find the fundamental system of solutions for the system:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; I = [1, M],\ M > 1.$$

(c) Find the constant k_1 and compute the first three successive approximations for the following system on the prescribed interval.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} \sin t & 0 \\ 0 & \cos t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ where } I = [0, \pi]$$

Q.4. Attempt Any Two: **14**

(a) (i) Prove that the general solution of the system $x' = Ax$, $t \in I$ is $x(t) = e^{tA}c$ where c is an arbitrary constant vector. Furthermore, prove that the solution of this system with the initial condition $x(t_0) = x_0$, $t_0 \in I$ is given by, $x(t) = e^{(t-t_0)A}x_0$, $t \in I$.

(ii) Prove that the fundamental matrix $\Phi(t)$ of the system $x' = Ax$, $t \in I$ where A is an n×n constant matrix, is given by $\Phi(t) = e^{tA}$ and the solution of the system $x' = Ax$; $t \in I$, $x(0) = E$; E is the identity matrix, is $x(t) = e^{tA}$.

(b) Consider the system $x' = Ax$, given that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Show that a fundamental matrix is $\Phi(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$.

Let $b(t) = \begin{bmatrix} \sin at \\ \cos bt \end{bmatrix}$. Find $\psi(t)$ of the non-homogeneous equation $x' = Ax + b(t)$ for which $\psi(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) Find a fundamental matrix for the system $x' = Ax$,

where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$.

Q.5. Attempt Any Two: **14**

(a) Let $a_1, a_2, \ldots$ be the positive zeros of the Bessel function $Jp(t)$. Then prove

that $\int_0^1 tJp(a_m t)Jp(a_n t)\,dt = \begin{cases} 0 & , m \neq n \\ \frac{1}{2}J_{p+1}^2(a_n) & , m = n \end{cases}$

(b) Determine e^{tA} for the system $x' = Ax$, where $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & -2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$

(c) Solve the following system of equations.

$x_1' = 5x_1 - 2x_2$

$x_2' = 2x_1 + x_2$